Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1. Introduction to Functions

7.2 One-to-One, Onto, Inverse functions

Watch this lecture and download the slides

Course Page: http://www.jarrar.info/courses/DMath/
More Online Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on (but not limited to) to chapter 7 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1 Introduction to Functions

In this lecture:

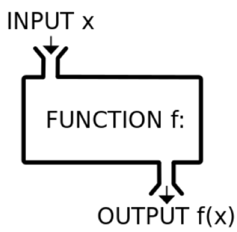
- ❑Part 1: **What is a function**
- ❑Part 2: Equality of Functions
- ❑Part 3: Examples of Functions
- ❑Part 3: Checking Well Defined Functions

Motivation

Many issues in life can be mathematized and used as functions:

- Div(x), mod(x), ….
- FatherOf(x), TruthTable (x)

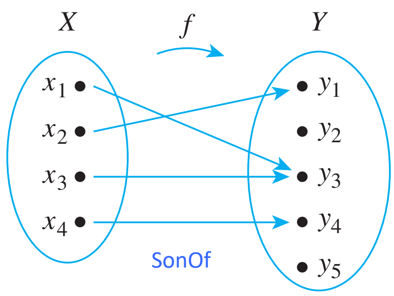
- In this lecture we focus on **discrete functions**



What is a Function

أبناء
Domain

اباء
Co-domain



علاقة بين عنصرين
كل عنصر في المجال يجب ان يكون
له صورة واحدة في المجال المقابل.
لا يوجد عنصر في المجال لا يوجد له
صورة في المجال المقابل

A function is a relation from X, the domain, to Y, the co-domain, that satisfies 2 properties: 1) Every element is related to some element in Y; 2) No element in X is related to more than one element in Y

,

5

Function Definition

• Definition

A **function** f **from a set** X **to a set** Y, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

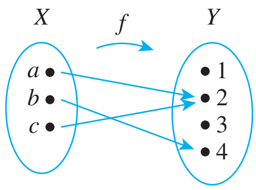
Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

6

Example

Let $X = \{a, b, c\}$ and $Y = \{1,2,3,4\}$. Define a function f from X to Y



a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

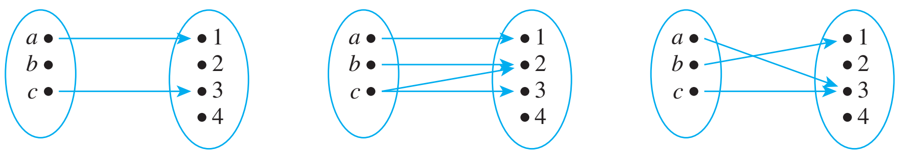
e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

7

Example

Which are functions?

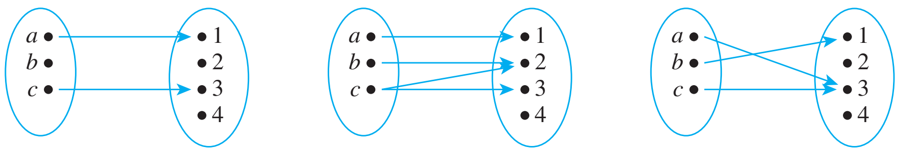


(a) (b) (c)

8

Example

Which are functions?



(a) (b) (c)

(a) b is not sent to any element in of Y
(b) The element c isn't sent to a unique element of Y
(c) Function

9

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1 Introduction to Functions

In this lecture:

- ❑Part 1: What is a function
- ❑Part 2: **Equality of Functions**
- ❑Part 3: Examples of Functions
- ❑Part 3: Checking Well Defined Functions

10

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x *in* J_3

$$f(x) = (x^2 + x + 1) \textit{ mod } 3 \quad \text{and } g(x) = (x + 2)^2 \textit{ mod } 3.$$

Does $f = g$?

x	x^2+x+1	$f(x) = (x^2+x+1) \textit{ mod } 3$	$(x+2)^2$	$g(x) = (x+2)^2 \textit{ mod } 3$
0	1	$1 \textit{ mod } 3 = 1$	4	$4 \textit{ mod } 3 = 1$
1	3	$3 \textit{ mod } 3 = 0$	9	$9 \textit{ mod } 3 = 0$
2	7	$7 \textit{ mod } 3 = 1$	16	$16 \textit{ mod } 3 = 1$

Equal functions

,

11

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let F: $\mathbf{R} \to \mathbf{R}$ and G: $\mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G$: $\mathbf{R} \to \mathbf{R}$ and $G + F$: $\mathbf{R} \to \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

$$\begin{aligned}(F+G)(x) &= F(x) + G(x) && \text{by definition of } F + G\\ &= G(x) + F(x) && \text{by the commutative law for addition of real numbers}\\ &= (G+F)(x) && \text{by definition of } G + F\end{aligned}$$

Hence $F + G = G + F$.

,

12

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1 Introduction to Functions

In this lecture:

- Part 1: What is a function
- Part 2: Equality of Functions
- Part 3: **Examples of Functions**
- Part 3: Checking Well Defined Functions

13

Examples of Functions

Identity Function

Function that always have the input is the same as the outputs, are called identity functions

Identity function send each element of X to the element that is identical to it.

$$I_X(x) = x \text{ for all } x \text{ in } X.$$

Examples of identity functions?

14

Examples of Functions

Sequences

An infinite sequence is a function defined on set of integers that are greater than or equal to a particular integer.

E.g., Define the following sequence as a function from the set of positive integers to the set of real numbers

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots$$

$f\colon \mathbf{Z}^{nonneg} \to \mathbf{R}$ $\qquad n \geq 0$

$$f(n) = \frac{(-1)^n}{n+1}$$

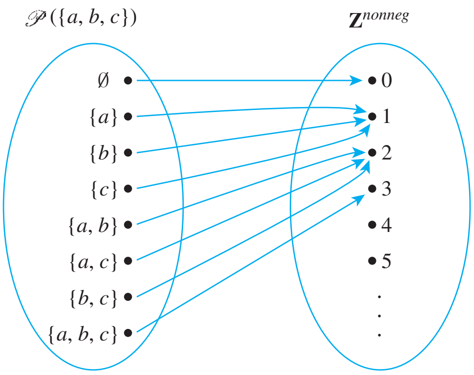
15

Examples of Functions

Function Defined on a Power Set

Draw an arrow diagram for F *as follows:*

$$F\colon \mathscr{P}(\{a, b, c\}) \to \mathbf{Z}^{nonneg}$$
$$F(X) = \text{the number of elements in } X.$$



16

Examples of Functions

Cartesian product

Define functions M: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and R: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$ as follows: For all ordered pairs (a, b) of integers,

$$M(a,b) = ab \quad \text{and} \quad R(a,b) = (-a,b).$$

M is the multiplication function that sends each pair of real numbers to the product of the two. R is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis.

Find the following:

a. $M(-1,-1)$ =1

b. M(½, ½) = ¼

c. M(√**2**, √**2**) =2

d. R(2,5) =(-2,5)

e. R(-2,5) =(2,5)

e. R(3,-4) =(-3,-4)

17

Examples of Functions

String Functions

g: S → **Z**

$g(s)$ = the number of a's in s.

Find the following.

a. $g(\epsilon)$ b. $g(bb)$ c. $g(ababb)$ d. $g(bbbaa)$

18

Examples of Functions

Logarithmic functions

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

- $\log_3 9$ = 2 because $3^2 = 9$.
- $\log_2 (1/2)$ = -1 because $2^{-1} = ½$.
- $\log_{10}(1)$ = 0 because $10^0 = 1$.
- $\log_2(2^m)$ = m because the exponent to which 2 must be raised to obtain 2^m is m.
- $2^{\log_2 m}$ = m because $\log_2 m$ is the exponent to which 2 must be raised to obtain m.

19

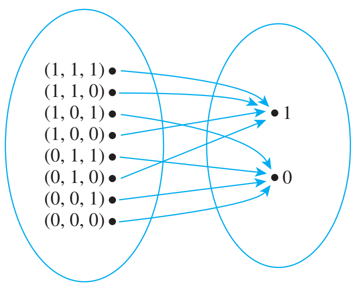
Examples of Functions

Boolean Functions

• Definition

An **(n-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0



20

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1 Introduction to Functions

In this lecture:

- Part 1: What is a function
- Part 2: Equality of Functions
- Part 3: Examples of Functions
- Part 3: **Checking Well Defined Functions**

Well-defined Functions

Checking Whether a Function Is Well Defined

A function is not well defined if it fails to satisfy at least one of the requirements of being a function

Example:

Define a function $f: \mathbf{R} \to \mathbf{R}$ by specifying that for all real numbers x, $f(x)$ is the real number y such that $x^2+y^2=1$.

There are two reasons why this function is not well defined: For almost all values of x either (1) there is no y that satisfies the given equation or (2) there are two different values of y that satisfy the equation

Consider when x=2
Consider when x=0

Well-defined Functions

Checking Whether a Function Is Well Defined

$f: \mathbf{Q} \rightarrow \mathbf{Z}$ defines this formula:

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Is f a well defined function?

No, Example:

$$f\left(\tfrac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\tfrac{3}{6}\right) = 3,$$

$$f\left(\tfrac{1}{2}\right) \neq f\left(\tfrac{3}{6}\right).$$

23

Well-defined Functions

Checking Whether a Function or not

Y= BortherOf(x)
Y= Parent Of(x)
Y= SonOf(x)
Y= FatherOf(x)
Y= Wife Of(x)
.
.
.

24